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# SCATTERING OF ANTIPLANE SHEAR WAVES BY A SUBMERGED CRACK

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**Abstract**—The scattering of time harmonic antiplane shear waves by a crack situated parallel to the surface of a homogenious elastic half-space is considered. The problem is investigated for both clamped and stress-free conditions at the surface of the half-space. Singular integral equations are derived for each case and treated numerically by the Gauss–Chebyshev technique. Harmonic stress intensity factors and crack opening displacements are computed as functions of dimensionless frequency, submerged depth-to-crack width ratio, and angle of incidence. Results are obtained for a moderately wide range of frequencies and are set out in graphical form.

## INTRODUCTION

The scattering of harmonic elastic waves by a line crack has been the subject of increased research activity. Although the subject has applications to problems in seismology and ultrasonics, recent interest has been related to problems of fracture. It is to this latter field of specialization that the present work is addressed.

Of particular interest in the mechanics of fracture is the effect of frequency, or inertia, on the stress intensity at the tips of a crack, since knowledge of this effect is required in order to predict the frequency at which crack spreading is likely to occur. The effect of frequency on stress intensity has been determined, to some extent, in a number of previous studies; those which are especially germane to the present study are listed in Refs. [1–3]. In the present study we consider, in addition, the effect of the submerged depth-to-crack width ratio on stress intensity. The effect is examined for both stress-free and clamped conditions at the surface of the half-space, and results are obtained for a moderately wide range of rrequencies.

## FORMULATION

Consider a crack of width $2l$ situated at a depth $h$ in a configuration parallel to the surface of a homogeneous elastic half-space (Fig. 1). Suppose the crack is excited by incident and reflected shear waves running at frequency $\omega$ so that waves of the same frequency are scattered. Let the displacements associated with the incident, reflected, and scattered waves be denoted as $w_i e^{-\omega t}$, $w_r\, e^{-i\omega t}$, and $w_s e^{-i\omega t}$, respectively. Hence the total displacement field is (with time factor discarded)

$$w_t = w_i + w_r + w_s. \tag{1}$$

Consistent with Fig. 1 the incident wave is described as

$$w_i = w_0\, e^{ik[x\cos\theta+(y-h)\sin\theta]} \tag{2}$$

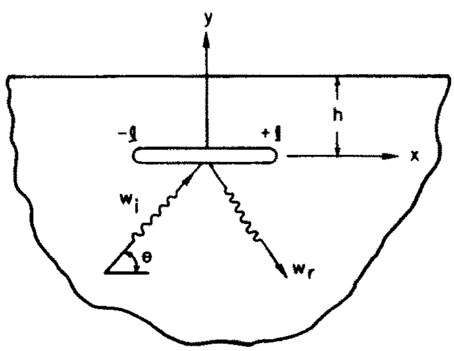


Fig. 1. Crack excited by incident and reflected waves.

where $\theta$ is the angle of incidence measured from the $x$ axis, and $k = \omega/c$, in which $c$ is the shear wave speed. The reflected wave is therefore of the form

$$w_r = \pm w_0 \, e^{ik[x \cos\theta - (y-h)\sin\theta]} \tag{3}$$

where + and − correspond, respectively, to stress-free and clamped conditions at the surface $y = h$.

We formulate the problem for the scattered displacement field $w$ (with subscript $s$ dropped). Thus, let $w^+$ and $w^-$ denote the displacements in regions $0 \le y \le h$ and $-\infty < y \le 0$ respectively, with $\tau_x^{\pm} = \mu w,_x^{\pm}$ and $\tau_y^{\pm} = \mu w,_y^{\pm}$ the associated stress components. For the clamped case the problem is specified as

$$(\nabla^2 + k^2) w^{\pm} = 0, \quad -\infty < x < \infty \tag{9}$$

with boundary conditions

$$w^+ = 0, \quad -\infty < x < \infty, \quad y = h \tag{5}$$

$$\tau_y^+ = \tau_y^-, \quad -\infty < x < \infty, \quad y = 0 \tag{6}$$

$$\tau_y^{\pm} = -2i\tau_0 \cos(kh \sin\theta) \, e^{ikx\cos\theta}, \quad |x| < l, \quad y = 0^{\pm} \tag{7}$$

$$w^+ - w^- = 0, \quad |x| > l, \quad y = 0 \tag{8}$$

where $\tau_0 = \mu w_0 k \sin\theta$. For the free surface case, conditions (7) and (8) are replaced by

$$\tau_y^+ = 0, \quad -\infty < x < \infty, \quad y = h \tag{9}$$

$$\tau_y^{\pm} = -2\tau_0 \sin(kh \sin\theta) \, e^{ikx\cos\theta}, \quad |x| < l, \quad y = 0^{\pm}. \tag{10}$$

In addition, the solution for each case must satisfy the usual edge condition at the crack tips.

## SOLUTION FOR CLAMPED CASE

Let $W^{\pm}(\xi, y)$ and $T_y^{\pm}(\xi, y)$ denote Fourier transforms of $w^{\pm}(x, y)$ and $\tau_y^{\pm}(x, y)$, respectively. It follows readily from eqn (4) and conditions (5) and (6) that

$$W^+ = \frac{T_y}{\alpha\mu} \operatorname{sech} \alpha h \sinh \alpha(h - y), \quad 0 \le y \le h \tag{11}$$

$$W^- = \frac{T_y}{\alpha\mu} e^{\alpha y}, \quad -\infty \le y \le 0 \tag{12}$$

where $T_y = T_y^+(\xi, 0) = T_y^-(\xi, 0)$, and where $\alpha = (\xi^2 - k^2)^{1/2}$ for $|\xi| > k$, with $\alpha = -i(k^2 - \xi^2)^{1/2}$ for $|\xi| < k$. From (11) and (12) we get the equation connecting transformed stress with transformed displacement discontinuity on the plane of the crack

$$2\frac{T_y}{\mu} = -(W^+ - W^-)\alpha(1 + e^{-2\alpha h}), \quad y = 0. \tag{13}$$

By inversion and use of conditions (7) and (8), there follows

$$L_x \phi(x) = -2i \, q_0/\mu \, e^{ikx\cos\theta}, \quad |x| < l$$

where

$$L_x = \frac{d^2}{dx^2} + k^2, \; q_0 = 2\tau_0 \cos(kh \sin\theta),$$

and

$$\phi(x) = \fint_{-l}^{l} v(s)H(s-x)\,\mathrm{d}s. \tag{15}$$

In (15)

$$v(x) = w^+(s) - w^-(s), \tag{16}$$

$$H(s-x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{i(s-x)\xi}\alpha^{-1}(1+\mathrm{e}^{-2\alpha h})\,\mathrm{d}\xi, \tag{17}$$

and $\fint_{-l}^{l}$ denotes principal value. Using results tabulated in reference [4] formula (17) may be replaced by

$$H(s-x) = \frac{i}{2}[H_0^{(1)}(k|s-x|) + H_0^{(1)}(kr)] \tag{18}$$

where $r = [(s-x)^2 + (2h)^2]^{1/2}$.

We next convert eqn (14) to a singular integral equation in the manner outlined by Noble [5], and adopted by Stone *et al.* [3] for an edge crack problem. Accordingly, integrating (14) gives

$$\phi(x) = A\,\mathrm{e}^{ikx} + B\,\mathrm{e}^{-ikx} + G(x), \tag{19}$$

where $A$ and $B$ are constants, and

$$G(x) = \frac{-2iq_0\,\mathrm{e}^{ikx\cos\theta}}{\mu k^2 \sin^2\theta}. \tag{20}$$

Further, upon differentiation and elimination of $A$ and $B$, there follows

$$\phi'(x) - \phi'(0)\cos kx + k\phi(0)\sin kx = G'(x) - G'(0)\cos kx + kG(0)\sin kx. \tag{21}$$

Substituting from (15), (18), and (20) yields the integral equation

$$\fint_{-l}^{l} v(s)kR(s,x)\,\mathrm{d}s = -\frac{4iq_0\psi(x)}{\mu k\sin^2\theta} \quad |x| < l \tag{22}$$

where

$$\psi(x) = \cos\theta[\cos(kx\cos\theta) - \cos kx] + i[\cos\theta\sin(kx\cos\theta) - \sin kx], \tag{23}$$

and

$$R(s,x) = H_1^{(1)}(k|s-x|)\,\mathrm{sgn}\,(s-x) + H_1^{(1)}(kr)\frac{s-r}{r}$$
$$- \cos kx\left[H_1^{(1)}(k|s|)\,\mathrm{sgn}s + H_1^{(1)}(kr_0)\frac{s}{r_0}\right] + \sin kx[H_0^{(1)}(k|s|) + H_0^{(1)}(kr_0)], \tag{24}$$

in which $r_0 = [s^2 + (2h)^2]^{1/2}$. It is noted that $R(s,x)$ has the character of a Cauchy-type kernel as $s \to x$ (or $k \to 0$.) Hence we may rewrite (22) with the singular part split off. Introducing the dimensionless quantities $x = l\bar{x}$, $s = l\bar{s}$, $\bar{k} = kl$, and $\delta = h/l$, and putting $v(s) = \bar{v}(\bar{s})$, $\bar{\tau}_0 = \tau_0 l$, and $\psi(x) = \bar{\psi}(\bar{x})$, we get, upon suppressing the bars

$$\frac{1}{\pi}\fint_{-1}^{1}\frac{v(s)}{s-x}\,\mathrm{d}s + \frac{ik}{4}\int_{-1}^{1} v(s)R^*(s,x)\,\mathrm{d}s = \frac{q_0\psi(x)}{\mu k\sin^2\theta}, \quad |x| < 1 \tag{25}$$

where

$$R^*(s,x) = R(s,x) + \frac{4i}{\pi k}(s-x)^{-1}. \quad (26)$$

The exact solution of eqn (25) may be easily obtained in the static case $k = 0$, if $\delta = 0$ or $\delta = \infty$. The appropriate solutions are

$$v(x) = \begin{cases} \frac{iq_0}{\mu}(1-x^2)^{1/2}, & \delta = 0 \\ \frac{2iq_0}{\mu}(1-x^2)^{1/2}, & \delta = \infty\ . \end{cases} \quad (27)$$

It is noted that the case $\delta = 0$ may be interpreted as that of a crack situated at $y = h$ in the extended infinite medium $-\infty < y < \infty$. At high frequencies eqn (25) may be expressed as

$$\frac{1}{\pi}\int_1^1 v(s)Q(s,x)\,ds = 4(2\pi)^{1/2}\frac{q_0\psi(x)\,e^{i\pi/4}}{\mu k^{3/2}\sin^2\theta} \quad (28)$$

where

$$Q(s,x) = \frac{e^{ik|s-x|}}{\sqrt{(|s-x|)}}\,\mathrm{sgn}\,(s-x) + \frac{e^{ikr}}{\sqrt{r}}\frac{s-x}{r} - \sin kx\left[\frac{e^{ik|s|}}{\sqrt{|s|}} + \frac{e^{ikr_0}}{\sqrt{r_0}}\right]. \quad (29)$$

Hence $|v| = 0(k^{-3/2})$ as $k \to \infty$ for all $\delta$.

At intermediate frequencies and depth-to-length ratios, eqn (25) may be treated numerically by the Gauss–Chebyshev approximation (see Erdogan and Gupta[6]). Since the crack displacement is expected to have a form similar to (27), we put

$$v(s) = \frac{iq_0}{\mu}u(s)(1-s^2)^{1/2}, \quad (30)$$

and assume $u(s)$ is bounded. Then use of the Gauss–Chebyshev integration formula[6]

$$\frac{1}{\pi}\int_{-1}^1 (1-s^2)^{1/2} f(s)\,ds \simeq \frac{1}{N+1}\sum(1-s_p^2)f_p,$$

where $f_p = f(s_p)$, and $s_p = \cos \pi p/N+1$, together with the Chebyshev collocation scheme gives

$$\frac{1}{N+1}\sum(1-s_p^2)u_p\left\{\frac{1}{s_p - x_q} + ik\frac{\pi}{4}R^*_{pq}\right\} = -\frac{i\psi_q}{k\sin^2\theta} \quad q = 1, 2, \ldots N \quad (31)$$

where $R^*_{pq}$ signifies $R^*(s_p, x_q)$, etc. and

$$x_q = \cos\frac{\mu v(x)}{2[2(l\mp x)]^{1/2}}.$$

As indicated by Erdogan and Gupta the system may be solved by choosing $N$ as an even integer and ignoring the equation corresponding to $q = (N/2)+1$.

Once the system is solved for $u_p$ the stress intensity (in terms of the original quantities), may be expressed as

$$K = \lim_{x\to\pm l}\frac{\mu v(x)}{2[2(l\mp x)]^{1/2}}. \quad (32)$$

Substituting from (30) and restoring original quantities gives

$$K = iq_0\sqrt{(l)}u(1)/2. \quad (33)$$

From the preceeding remarks we note that $|u(1)| = 0(1)$ as $k \to 0$, and $|u(1)| = 0(k^{-3/2})$ as $k \to \infty$, for all $\delta$.

## SOLUTION FOR FREE EDGE CASE

The integral equation for the free edge case follows by the same line of reasoning as for the clamped edge case. Thus, omitting repetitive details, the result may be expressed as

$$\frac{1}{\pi}\int_{-1}^{1} \frac{v(s)}{s-x}\,\mathrm{d}s + \frac{ik}{4}\int_{-1}^{1} v(s)T^*(s,x)\,\mathrm{d}s = \frac{-ip_0\psi(x)}{\mu k \sin^2\theta}, \quad |x|<1 \tag{34}$$

where

$$p_0 = 2\tau_0 \sin(k\delta \sin\theta), T^*(s,x) = T(s,x) + \frac{4i}{\pi k}(s-x)^{-1},$$

$$T(s,x) = H_1^{(1)}(k|s-x|)\,\mathrm{sgn}\,(s-x) - H_1^{(1)}(kr)\frac{s-x}{r}$$

$$-\cos kx\left[H_1^{(1)}(k|s|)\,\mathrm{sgn}s - H_1^{(1)}(kr_0)\frac{s}{r_0}\right] + \sin kx[H_0^{(1)}(k|s|) - H_0^{(1)}(kr_0)], \tag{35}$$

and where $\psi(x)$ is given by (23). Again, using (30) with $q_0$ replaced by $p_0$, and applying the Gauss–Chebyshev approximation yields

$$\frac{1}{N+1}\sum_{1}^{N}(1-s_p^2)u_p\left\{\frac{1}{s_p-x_q} + \frac{ik\pi}{4}T^*_{pq}\right\} = \frac{-\psi_q}{k\sin^2\theta}, \quad q = 1,\dots N. \tag{36}$$

Moreover, applying formula (32) gives

$$K = ip_0\sqrt{(l)}u(1)/2. \tag{37}$$

For this case $u(1)$ can be shown to exhibit the same asymptotic behavior as in the clamped case.

## RESULTS AND DISCUSSION

Algebraic systems (31) and (36) were solved by digital computer for values of dimensionless frequency in the range $0 \le k \le 5$, and for selected values of submerged depth-to-half crack width ratio $\delta$ and angle of incidence $\theta$. We found that a value of $N = 20$ gave sufficient accuracy (an average difference of less than 1% by random check with $N = 30$), for the range of frequencies considered. Figures 2 and 3 depict the normalized stress intensity as a function of

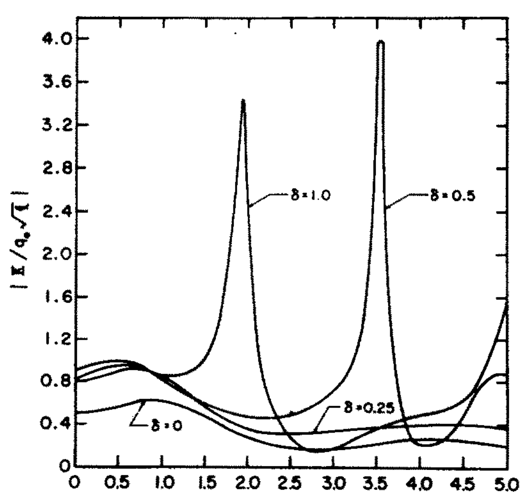


Fig. 2. Normalized stress intensity vs dimensionless frequency, clamped condition, $\delta = 0, 0.25, 0.5, 1.0$.

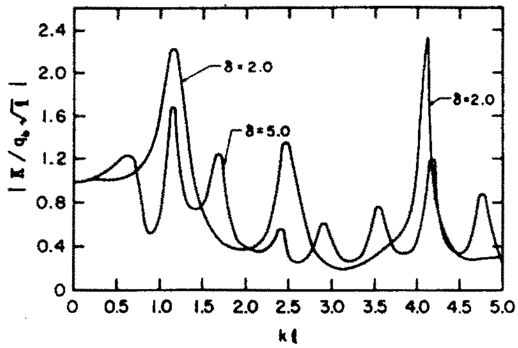


Fig. 3. Normalized stress intensity vs dimensionless frequency, clamped condition, $\delta = 2.0, 5.0$.

frequency for the clamped case, when $\theta = 90°$. The curve marked $\delta = 0$ agrees closely with the results of Loeber and Sih[1] and Mal[2], apart from a factor of 2 due to the reflected wave. At $\delta = 5$ the low frequency results are nearly equal to those given in references [1] and [2], as they should. (As $\delta \to \infty$ the results should agree for all frequencies.) At intermediate values of $\delta$ peaks occur at frequencies which are reasonably close to the resonant frequencies $k_n l = (n\pi/2\delta)$, $n = 1, 3, 5, \ldots$, of an equivalent elastic layer of infinite length. This suggests a quasi-resonant condition in the layer between the crack and the boundary. This condition is apparently characterized by the increasingly higher peaks at high frequencies as the crack approaches the boundary. As previously shown, however, $|K/q_0\sqrt{l}| = 0(k^{-3/2})$ as $k \to \infty$ for any $\delta$. Hence it is expected that the maximum peak will occur at a moderate frequency. We infer from the results that the maximum peak will occur in the range $3.5 < k < 6.5$ at a depth-to-width ratio in the range $0.25 < \delta < 0.5$.

Figures 4 and 5 illustrate the corresponding results for the free edge case. Since this case

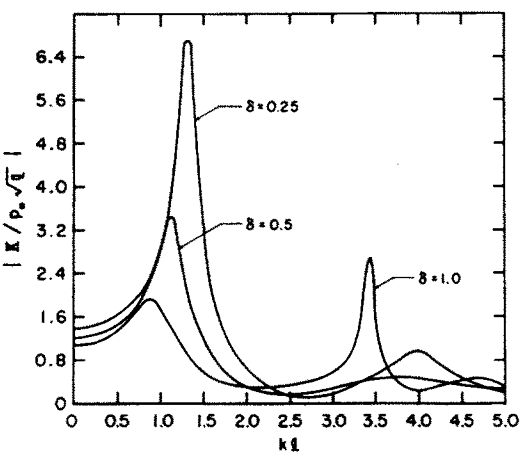


Fig. 4. Normalized stress intensity vs dimensionless frequency, free condition, $\delta = 0.25, 0.5, 1.0$.

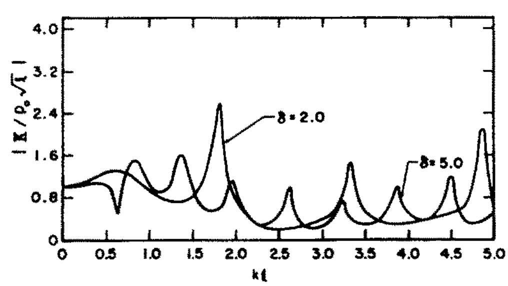


Fig. 5. Normalized stress intensity vs dimensionless frequency, free condition, $\delta = 2.0, 5.0$.

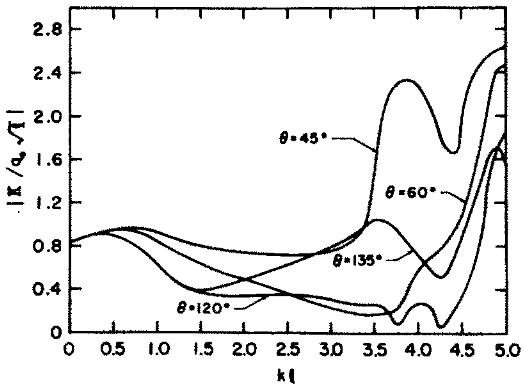


Fig. 6. Normalized stress intensity vs dimensionless frequency, clamped condition, $\delta = 0.5$, $\theta = 45°, 60°, 120°, 135°$.

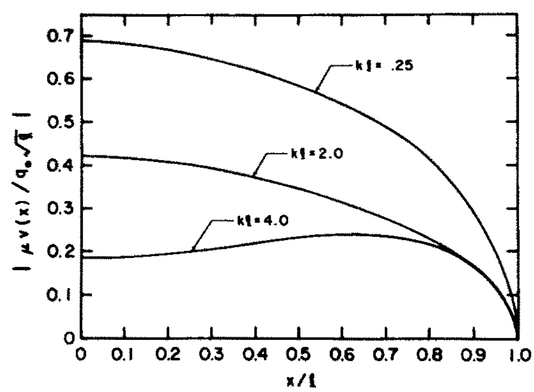


Fig. 7. Normalized crack opening displacement vs distance along crack, clamped condition, $\delta = 0.25$.

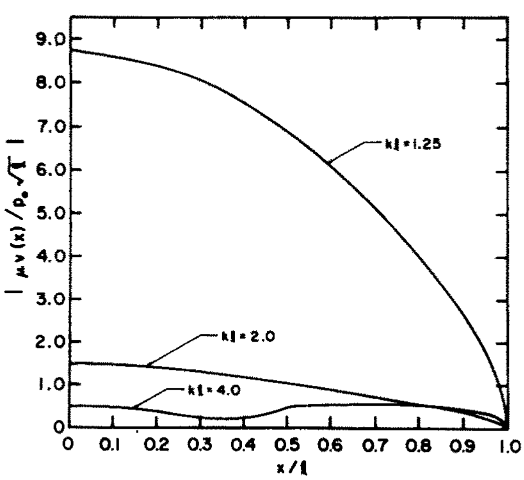


Fig. 8. Normalized crack opening displacement vs distance along crack, free condition, $\delta = 0.25$.

has no static counterpart, no comparison can be made with static results. By comparison with the clamped case a similar quasi-resonant effect occurs near the frequencies $k_m l = m\pi/\delta$, $m = 1, 2, 3, \ldots$. This effect is illustrated only for the higher values of $\delta$. In this case, however, an additional low-frequency response of appreciable magnitude occurs when the crack approaches the boundary. This response is due evidently to the absence of boundary restraint. Since the stress intensity is of the same order of magnitude at high frequencies as for the clamped case, the results suggest that the maximum peak intensity will occur at a lower frequency than the clamped case, and at a value of $\delta$ in the reduced range $0 < \delta < 0.25$.

The effect of angle of incidence on stress intensity is shown in Fig. 6 for the clamped case when $\delta = 0.5$. It is observed that the stress intensity is virtually unaffected by angle of incidence at low frequencies, but significant differences occur at moderately high frequencies. A high frequency attenuation, similar to that for normal incidence is predicted for these curves.

Figures 7 and 8 depict the crack opening displacement for the case of normal incidence when $\delta = 0.25$. The curve marked $kl = 1.25$ in Fig. 8 illustrates the quasi-resonant effect alluded to above. Finally, the rippling effect at higher frequencies is similar to that obtained by Mal[2] for a crack in an infinite medium.

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